Hierarchy of classicality indicators for N-level systems

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1. Introduction

2. Stratification by group action

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The Wigner function

Wigner quasiprobability distribution: $W(\Omega_N) = \mathrm{tr}\left[\varrho\, \Delta(\Omega)\right]$,

- density matrix $\varrho \in \mathfrak{P}_N$: $\quad \varrho = \varrho^\dagger\,, \quad \varrho \geq 0\,, \quad \mathrm{tr}\,(\varrho) = 1\,,$
- Stratonovich-Weyl kernel $\Delta(\Omega) \in \mathfrak{P}^*_N$: $\Delta = \Delta^\dagger\,,\ \ \mathrm{tr}\,(\Delta) = 1\,,\ \mathrm{tr}\,(\Delta^2) = N\,,$
- phase space $\Omega_N \ \rightarrow\ \mathbb{F}^N_{d_1,d_2,\dots,d_s} = U(N)/H$ which is a complex flag manifold with isotropy group $H = U(k_1) \times \cdots \times U(k_{s+1}) \in U(N)$, where $k_1 = d_1\,, \quad k_{i+1} = d_{i+1} - d_i\,, \quad d_{s+1} = N\,, \quad d_i \in \mathbb{Z}^+\,.$

Aim of the talk:

To introduce a family of measures of classicality of a given system associated to a natural stratification of state space $\mathfrak{P}_N$ by the unitary orbit types and to evaluate the rate of classicality for low-dimensional quantum systems.

Family of the Wigner functions:

$$W_{\boldsymbol{\xi}}^{(\boldsymbol{\nu})}(\Omega_N) = \frac{1}{N}\left(1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\xi})\right) ,$$

- $\boldsymbol{\xi}$ is (N^2-1)-dimensional Bloch vector,
- parameter $\boldsymbol{\nu} = (\nu_1 , \cdots , \nu_{N-2})$ labels members of WF family,
- vector $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \mu_8 \boldsymbol{n}^{(8)} + \ldots + \mu_{N^2-1} \boldsymbol{n}^{(N^2-1)}$,
- orthonormal vectors $\boldsymbol{n}_\mu^{(s^2-1)} = \frac{1}{2}\operatorname{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right)$,
- $\mathfrak{su}(N)$ algebra orthonormal Hermitian basis $\boldsymbol{\lambda} = \{\lambda_1 , \cdots , \lambda_{N^2-1}\}$,
- real coefficients $\mu_3^2 + \mu_8^2 + \cdots + \mu_{s^2-1}^2 = 1 , \quad s = \overline{2, N}$.

Wigner function is non-negative, $W_{\boldsymbol{\xi}}^{(\boldsymbol{\nu})}(\Omega_N) \geq 0$, for any state with

$$0 \leq \boldsymbol{\xi}^2 \leq r_*^2(N) , \quad \text{where} \quad r_*(N) = \sqrt{N+1}/(N^2-1) .$$

Nonclassicality characteristics of states

Nonclassicality measures based on the violation of the Wigner function semi-positivity can be divided into different types:

1. (Nonclassical distance) based on a **distance** of a state from the "classical states":

$$\delta_\varrho = \inf_{x \in \mathfrak{P}_{\mathrm{Cl}}} D(\varrho, x)\,,$$

   where states with positive Wigner functions are taken as the reference "classical states", $\mathfrak{P}_{\mathrm{Cl}}$.

2. (Kenfack-Życzkowski indicator) based on the **volume of the negative part** of the Wigner function:

$$\delta_N = \int_{\Omega_N} \mathrm{d}\Omega_N \left| W(\Omega_N) \right| - 1\,.$$

Qubit nonclassical distance and KZ-indicator

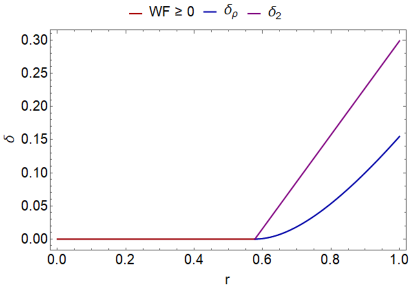
Qubit Wigner function: $W_{\xi}(\Omega_2) = \frac{1}{2}\left(1 + \sqrt{3}\,\xi \cdot \boldsymbol{n}\right)$.

Qubit nonclassicality distance for Hilbert-Schmidt metric:

$$\delta_{\varrho} = \theta[\mathrm{r} - \tfrac{1}{\sqrt{3}}]\left(\frac{\mathrm{r}}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right).$$

Qubit KZ-indicator:

$$\delta_2 = \theta[\mathrm{r} - \tfrac{1}{\sqrt{3}}]\left(\frac{3\mathrm{r}^2+1}{2\sqrt{3\mathrm{r}}} - 1\right).$$



The global indicator of classicality

3. (Global indicator of classicality) as the **relative volume** of a subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ of the state space $\mathfrak{P}_N$, consisting of states whose Wigner functions are **positive**.

The global indicator of classicality of states over a given stratum:

$$\mathcal{Q}_N[H_\alpha] = \frac{\mathrm{Vol}\left(\mathfrak{P}^{(+)}_{[H_\alpha]}\right)}{\mathrm{Vol}\left(\mathfrak{P}_{[H_\alpha]}\right)} . \tag{1}$$

Due to the $SU(N)$-covariance of the Wigner function, the relative volume (1) is equal to the corresponding ratio of volumes in the orbit space $\mathcal{O}[\mathfrak{P}_N]$, i.e., the volumes of images of the quotient projection of state space $\mathfrak{P}_N$ to the unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$:

$$\pi : \ \mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] = \mathfrak{P}_N/SU(N) .$$

Definition 1. The unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$ is the quotient space under the equivalence relation imposed by the adjoint $SU(N)$-action on the state space $\mathfrak{P}_N$ with quotient (canonical) mapping $\pi : \mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] = \mathfrak{P}_N/SU(N)$.

Definition 2. The set $\Omega_N^{(+)}[\varrho] = \{ x \in \Omega_N \mid W_\varrho(\Omega_N) \geq 0 \}$ is a subset of phase space Ω_N where the Wigner function of a given state ϱ is non-negative.

Definition 3. The subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ is composed from states $\varrho \in \mathfrak{P}_N : \Omega_N^{(+)}[\varrho] = \Omega_N$.

Definition 4. The subset $\mathcal{O}[\mathfrak{P}_N^{(+)}] = \pi[\mathfrak{P}_N^{(+)}] = \{\pi(x) \mid x \in \mathfrak{P}_N^{(+)} \}$ represents the image of $\mathfrak{P}_N^{(+)}$ under the quotient mapping π.

Orbit space of $SU(N)$ group adjoint action on state space

The orbit space $\mathcal{O}[\mathfrak{P}_N]$ is an ordered $(N-1)$-simplex in the space of eigenvalues $\boldsymbol{r} = \{r_1, \dots, r_N\}$ of a density matrix $\varrho = U \varrho_{diag} U^\dagger$:

$$\mathcal{O}[\mathfrak{P}_N] = \{ \, \boldsymbol{r} \in \mathbb{R}^N \mid \sum_{i=1}^{N} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq \dots \geq r_{N-1} \geq r_N \geq 0 \, \} .$$

The subspace $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ is a dual cone $(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) = r_1\pi_N + \dots + r_N\pi_1$ of a subset $\mathcal{O}[\mathfrak{P}_N] \subset \mathbb{R}^{N-1}$, $\boldsymbol{\pi} = \{\pi_1, \dots, \pi_N\}$ are SW kernel eigenvalues,

$$\mathcal{O}[\mathfrak{P}_N^{(+)}] = \{ \, \boldsymbol{\pi} \in \mathbf{spec}\,(\Delta(\Omega_N)) \mid (\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) \geq 0, \quad \forall\, \boldsymbol{r} \in \mathcal{O}[\mathfrak{P}_N] \, \} .$$

The lower bound of Wigner function determines the positivity region:

$$W_N^{(-)} = \sum_{i=1}^{N} \pi_i r_{N-i+1} \leq W(\Omega_N) \leq \sum_{i=1}^{N} \pi_i r_i = W_N^{(+)} .$$

State space $\mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad X \geq 0\,, \quad \operatorname{tr}(X) = 1\}$

The unitary $U(N)$ automorphism of the Hilbert space of an N-level quantum system induces the adjoint $SU(N)$-action on the state space:

$$g \cdot \varrho = g\, \varrho\, g^\dagger\,, \qquad g \in SU(N)\,,$$

which sets equivalence relations between elements of $\mathfrak{P}_N$ and gives rise to its decomposition over the strata:

$$\mathfrak{P}_{[H_\alpha]} := \{x \in \mathfrak{P}_N \mid H_x \text{ is conjugate to } H_\alpha\}\,, \quad \mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]}\,.$$

A subgroup $H_x \subset SU(N)$ is the isotropy group of a point $x \in \mathfrak{P}_N$,

$$H_x = \{g \in SU(N) \mid g \cdot x = x\}\,,$$

and points $x\,, y \in \mathfrak{P}_N$ are said to be of the same type if their stabilizers H_x and H_y are conjugate subgroups of $SU(N)$ group.

The Hilbert-Schmidt metric

The Hilbert–Schmidt volume element for general case:

$$dV_{HS} = \sqrt{N} \prod_{j=1}^{N-1} d\Lambda_j \prod_{j<k}^{1\dots N} (\Lambda_j - \Lambda_k)^2 \left| \prod_{j<k}^{1\dots N} 2\,\mathrm{Re}(U^{-1}dU)_{jk}\, \mathrm{Im}(U^{-1}dU)_{jk} \right| .$$

The Hilbert–Schmidt volume element for degenerate cases:

$$dV_{HS} = \sqrt{\frac{N}{k}} \prod_{i=1}^{m-1} d\Lambda_i dr \prod_{i<j}^{1\dots m} (\Lambda_i - \Lambda_j)^2 \prod_{i=1}^{m} (\Lambda_i - r)^{2k} \times$$

$$\prod_{i<j}^{1\dots m} |2\,\mathrm{Re}(\Omega_{ij})\,\mathrm{Im}(\Omega_{ij})| \prod_{i=1}^{m} \prod_{j=m+1}^{N} |2\,\mathrm{Re}(\Omega_{ij})\,\mathrm{Im}(\Omega_{ij})| .$$

Quantumness of a single qutrit

Qutrit density matrix $\varrho_3 = \frac{1}{3}(I + \sqrt{3}\sum_{\nu=1}^{8}\xi_\nu\lambda_\nu)$ has the spectrum $r_{1,2} = 1/3\,(1 \pm \sqrt{3}\xi_3 + \xi_8)$ with $1 \geq r_1 \geq r_2 \geq r_3 \geq 0$.

Qutrit SW kernel $\Delta = U\,\frac{1}{3}(I + 2\sqrt{3}(\mu_3\lambda_3 + \mu_8\lambda_8))\,U^\dagger$, with $\mu_3 = \sin\zeta$, $\mu_8 = \cos\zeta$, $\zeta \in [0, \pi/3]$, has the spectrum $\pi_{1,2} = 1/3\,(1 \pm 2\sqrt{3}\sin\zeta + 2\cos\zeta)$ with $\pi_1 \geq \pi_2 \geq \pi_3$.

Qutrit orbit space and its subspace of WF positivity are respectively

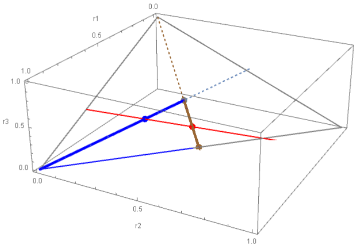
$$\mathcal{O}[\mathfrak{P}_3] : \ \left\{ \boldsymbol{r} \in \mathbb{R}^2 \ \middle| \ \sum_{i=1}^{3} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq r_3 \geq 0 \right\},$$

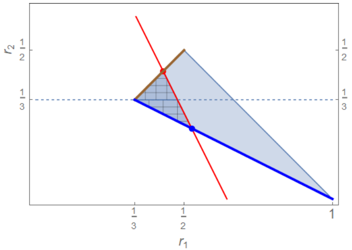
$$\mathcal{O}[\mathfrak{P}_3^{(+)}] : \left\{ \zeta \in [0, \pi/3] \ \middle| \ r_3 \geq \frac{r_1(4\cos\zeta - 1) - r_2\left(1 + 2\cos\zeta - 2\sqrt{3}\sin\zeta\right)}{1 + 2\cos\zeta + 2\sqrt{3}\sin\zeta} \right\}.$$

Strata of a qutrit orbit space:

1. stratum $\mathcal{O}_{123}$ of $\dim(\mathcal{O}) = 6$, $r_1 \neq r_2 \neq r_3$, for regular orbits,
2. two strata $\mathcal{O}_{1|23}$ and $\mathcal{O}_{12|3}$, $r_1 \neq r_2 = r_3$ and $r_1 = r_2 \neq r_3$, with $\dim(\mathcal{O}_{1|23}) = \dim(\mathcal{O}_{12|3}) = 4$,
3. stratum $\mathcal{O}_0$ of $\dim(\mathcal{O}_0) = 0$, $r_1 = r_2 = r_3$, for the orbit of maximally mixed state.

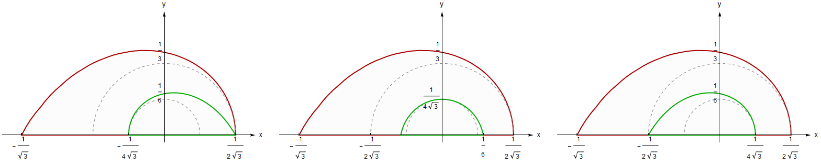
Wigner function lower bound positivity region:





Under transformation $\xi_3 = \sqrt{3}\,\mathrm{r} \sin\left(\frac{\varphi}{3}\right)$, $\xi_8 = \sqrt{3}\,\mathrm{r} \cos\left(\frac{\varphi}{3}\right)$, qutrit orbit space maps to the upper half-plane outlined by the Maclaurin trisectrix (for $x = \mathrm{r}\cos\varphi$, $y = \mathrm{r}\sin\varphi$): $\mathrm{r}(\varphi, 1/\sqrt{3}) = \frac{1}{2\sqrt{3}\cos(\varphi/3)}$.

The state space of a qutrit is divided into bands:



$\zeta = 0$ $\zeta = \pi/6$ $\zeta = \pi/3$

- the Wigner function is always positive for $\mathrm{r} \in [0, \frac{1}{4\sqrt{3}}]$,
- necessarily has some negative values for $\mathrm{r} \in [\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}]$,
- positive for the specific choice of the kernel for $\mathrm{r} \in [\frac{1}{4\sqrt{3}}, \frac{1}{2\sqrt{3}}]$.

Maximal stratum $\mathcal{Q}$-indicator:

$$\mathcal{Q}_3[\mathrm{g}_{\mathrm{HS}}] = \frac{20\cos^2(\zeta-\pi/6)+1}{128(4\cos^2(\zeta-\pi/6)-1)^5}\,,$$

Non-maximal stratum $\mathcal{Q}$-indicator:

$$\mathcal{Q}_3^{12|3}[\mathrm{g}_{\mathrm{HS}}] = \frac{1}{32}\csc^5\left(\frac{\pi}{6}+\zeta\right),$$

$$\mathcal{Q}_3^{1|23}[\mathrm{g}_{\mathrm{HS}}] = \frac{\sec^5(\zeta)}{1024}\,.$$

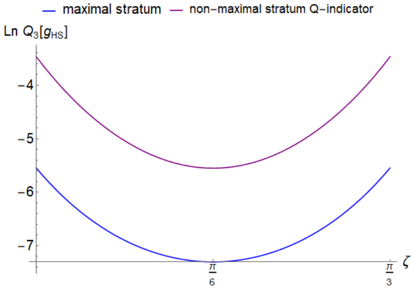


Figure 1: General $\mathcal{Q}$-indicator given by the solid blue curve; the sum of degenerate $\mathcal{Q}$-indicators given in solid purple curve.

CONJECTURE: more symmetry – more classicality!

Let us arrange the isotropy groups H_α in ascending order, starting from the maximal torus T_N up to the whole group $SU(N)$ [a],

$$T_N = H_{\min} < H_1 < \cdots < H_{\max} = SU(N)\,.$$

Then

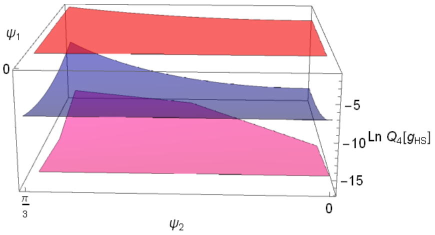
$$\mathcal{Q}_N[T_N] < \mathcal{Q}_N[H_1] < \cdots < \mathcal{Q}_N[SU(N)] = 1\,.$$

[a] If H and K are isotropy subgroups of G, we define a partial ordering on equivalence classes by writing $(H) < (K)$ if H is G-conjugate to a subgroup of K. This defines a partial ordering on the set of isotropy types.

Non-maximal stratum $\mathcal{Q}$-indicators of a qubit-qubit system

Qubit-qubit global indicators for stratum of orbits whose isotropy groups are:

- $SU(3) \times U(1)$ (red domain) for $\mathcal{Q}_4^{12|3|4\,,\,1|2|34}$;
- $SU(2) \times SU(2)$ (blue domain) for $\mathcal{Q}_4^{1|23|4}$;
- $SU(2)$ (magenta domain) for $\mathcal{Q}_4^{123|4\,,\,12|34\,,\,1|234}$.



Summary

- A true classicality measure, being universal for different quasi-probability representations, may be sensitive to the geometry of a state space.
- To recover the symmetry of the global indicator of classicality due to an interplay of symmetries of the chosen metric and moduli space of parameters, one should consider not merely the "degeneracy" orbit stratification of the state space, but unify the orbits with the same dimension.
- The conjecture holds for the considered cases but is to be proved for an arbitrary N.

Thank you!